Integrals of Complex-Valued Functions

Definition, Basic Properties, and the Integral Inequality

Lecture Notes

October 22, 2026

1 Integral of a Complex-Valued Function

Definition

For $f : [a, b] \to \mathbb{C}$, $f(t) = x(t) + iy(t)$,

$$\int_a^b f(t)\,dt := \int_a^b x(t)\,dt + i\int_a^b y(t)\,dt.$$

Basic properties.

- **Linear:** $\displaystyle\int_a^b (\alpha f(t) + \beta g(t))\,dt = \alpha\int_a^b f(t)\,dt + \beta\int_a^b g(t)\,dt \quad (\alpha, \beta \in \mathbb{C}).$
- **Additive:** $\displaystyle\int_a^b f(t)\,dt + \int_b^c f(t)\,dt = \int_a^c f(t)\,dt.$
- **Change of variables:** if $t \mapsto g(t)$ is piecewise differentiable and increasing on $[c, d]$ with $g(c) = a$, $g(d) = b$, then $\displaystyle\int_a^b f(t)\,dt = \int_c^d f(g(s))g'(s)\,ds.$
- **Change of orientation:** $\displaystyle\int_b^a f(t)\,dt = -\int_a^b f(t)\,dt.$

Lemma

$$\left|\int_a^b f(t)\,dt\right| \le \int_a^b |f(t)|\,dt.$$

Remark. For real-valued $f : [a, b] \to \mathbb{R}$ this would just be the observation $-|f| \le f \le |f|$; the complex case needs a different trick.

Proof (the "change of phase" trick). Let $M := \int_a^b f(t)\,dt$. If $M = 0$ the inequality is obvious; assume $M \neq 0$. Then

$$|M|^2 = \mathrm{Re}\left(\bar{M}\int_a^b f(t)\,dt\right) = \int_a^b \mathrm{Re}\left(\bar{M}f(t)\right)dt$$
$$\le \int_a^b |\bar{M}||f(t)|\,dt = |M|\int_a^b |f(t)|\,dt.$$

using that $|M|^2$ is real, and $\mathrm{Re}(w) \le |w|$. Dividing by $|M| \neq 0$ gives $|M| \le \int_a^b |f(t)|\,dt$. □